Quantum numbers and Angular Momentum Algebra Morten Hjorth-Jensen¹ National Superconducting Cyclotron Laboratory and Department of Physics and Astronomy, Michigan State University, East Lansing, MI 48824, USA¹ Spring 2016

Quantum numbers

Outline

- Discussion of single-particle and two-particle quantum numbers, uncoupled and coupled schemes
- Discussion of angular momentum recouplings and the Wigner-Eckart theorem
- Applications to specific operators like the nuclear two-body tensor force

For quantum numbers, chapter 1 on angular momentum and chapter 5 of Suhonen and chapters 5, 12 and 13 of Alex Brown. For a discussion of isospin, see for example Alex Brown's lecture notes chapter 12, 13 and 19.

Motivation

When solving the Hartree-Fock project using a nucleon-nucleon interaction in an uncoupled basis (m-scheme), we found a high level of degeneracy. One sees clear from the table here that we have a degeneracy in the angular momentum j, resulting in 2j + 1 states with the same energy. This reflects the rotational symmetry and spin symmetry of the nuclear forces.

Quantum numbers	Energy [MeV]
$0s_{1/2}^{\pi}$	-40.4602
$0s_{1/2}^{\pi}$	-40.4602
$0s_{1/2}^{\nu}$	-40.6426
$0s_{1/2}^{\nu}$	-40.6426
$0p_{1/2}^{\pi}$	-6.7133
$0p_{1/2}^{\pi}$	-6.7133
$0p_{1/2}^{\nu'}$	-6.8403
$0p_{1/2}^{\nu'}$	-6.8403
$0p_{3/2}^{\pi'}$	-11.5886
$0p_{3/2}^{\pi'}$	-11.5886
$0p_{3/2}^{\pi'}$	-11.5886
$0p_{3/2}^{\pi}$	-11.5886
$0p_{3/2}^{\nu}$	-11.7201

Single-particle and two-particle quantum numbers

In order to understand the basics of the nucleon-nucleon interaction and the pertaining symmetries, we need to define the relevant quantum numbers and how we build up a single-particle state and a two-body state, and obviously our final holy grail, a many-boyd state.

- For the single-particle states, due to the fact that we have the spin-orbit force, the quantum numbers for the projection of orbital momentum *I*, that is *m_i*, and for spin *s*, that is *m_s*, are no longer so-called good quantum numbers. The total angular momentum *j* and its projection *m_j* are then so-called *good* quantum numbers.
- $\bullet\,$ This means that the operator \hat{J}^2 does not commute with \hat{L}_z or $\hat{S}_z.$
- We also start normally with single-particle state functions defined using say the harmonic oscillator. For these functions, we have no explicit dependence on j. How can we introduce single-particle wave functions which have j and its projection

Single-particle and two-particle quantum numbers, brief review on angular momenta etc

We have that the operators for the orbital momentum are given by

$$L_{x} = -i\hbar(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}) = yp_{z} - zp_{y},$$
$$L_{y} = -i\hbar(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}) = zp_{x} - xp_{z},$$

$L_z = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) = xp_y - yp_x.$

Single-particle and two-particle quantum numbers, brief review on angular momenta etc

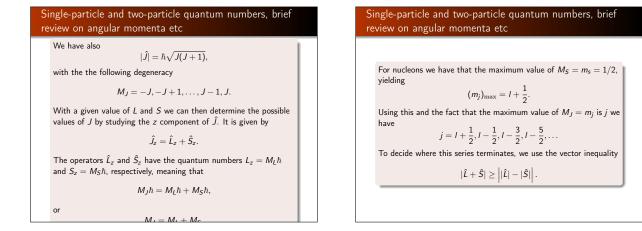
Since we have a spin orbit force which is strong, it is easy to show that the total angular momentum operator

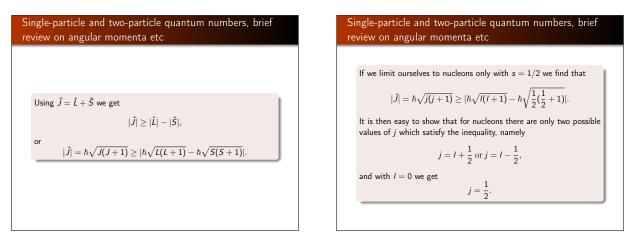
$$\hat{J} = \hat{L} + \hat{S}$$

does not commute with \hat{L}_z and $\hat{S}_z.$ To see this, we calculate for example

$$\begin{split} [\hat{L}_{z}, \hat{J}^{2}] &= [\hat{L}_{z}, (\hat{L} + \hat{S})^{2}] & (1) \\ &= [\hat{L}_{z}, \hat{L}^{2} + \hat{S}^{2} + 2\hat{L}\hat{S}] \\ &= [\hat{L}_{z}, \hat{L}\hat{S}] = [\hat{L}_{z}, \hat{L}_{x}\hat{S}_{x} + \hat{L}_{y}\hat{S}_{y} + \hat{L}_{z}\hat{S}_{z}] \neq 0, \end{split}$$

since we have that $[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$ and $[\hat{L}_z, \hat{L}_y] = i\hbar \hat{L}_x$.





Single-particle and two-particle quantum numbers, brief review on angular momenta etc

Let us study some selected examples. We need also to keep in mind that parity is conserved. The strong and electromagnetic Hamiltonians conserve parity. Thus the eigenstates can be broken down into two classes of states labeled by their parity $\pi = +1$ or $\pi = -1$. The nuclear interactions do not mix states with different parity.

For nuclear structure the total parity originates from the intrinsic parity of the nucleon which is $\pi_{\rm intrinsic}=+1$ and the parities associated with the orbital angular momenta $\pi_l=(-1)^l$. The total parity is the product over all nucleons

 $\pi = \prod_{i} \pi_{\text{intrinsic}}(i) \pi_{i}(i) = \prod_{i} (-1)^{l_{i}}$

The basis states we deal with are constructed so that they conserve parity and have thus a definite parity.

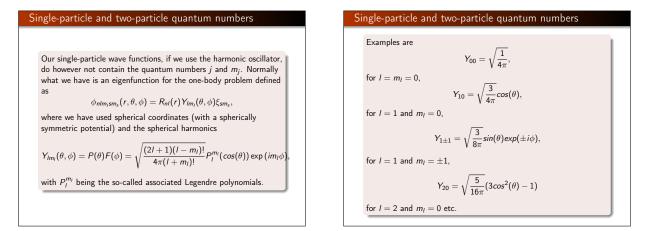
Note that we do have parity violating processes, more on this later although our focus will be mainly on non-parity viloating processes

Single-particle and two-particle quantum numbers

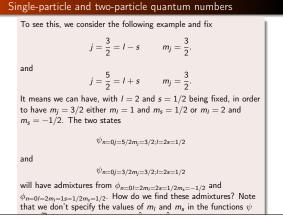
Consider now the single-particle orbits of the 1s0d shell. For a 0d state we have the quantum numbers l = 2, $m_l = -2, -1, 0, 1, 2$, s + 1/2, $m_s = \pm 1/2$, n = 0 (the number of nodes of the wave function). This means that we have positive parity and

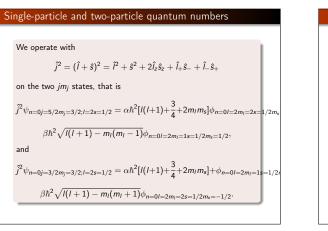
$$j = \frac{3}{2} = l - s \qquad m_j = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}.$$

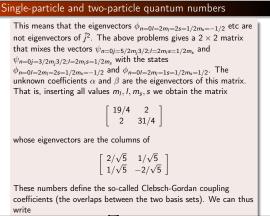
and
$$j = \frac{5}{2} = l + s \qquad m_j = -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}.$$

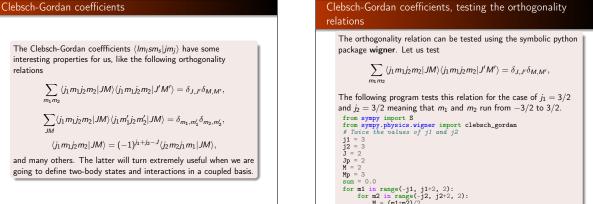


Single-particle and two-particle quantum numbers		Single-pa	
		To see t	
How can we get a function in terms of j and m_j ? Define now $\phi_{nlm_lsm_s}(r,\theta,\phi)=R_{nl}(r)Y_{lm_l}(\theta,\phi)\xi_{sm_s},$		and	
and $\begin{split} \psi_{njm_j;lm_lsm_s}(r,\theta,\phi), \\ \text{as the state with quantum numbers } jm_j. \text{ Operating with} \\ \hat{j}^2 = (\hat{l}+\hat{s})^2 = \hat{l}^2 + \hat{s}^2 + 2\hat{l}_z\hat{s}_z + \hat{l}_+\hat{s} + \hat{l}\hat{s}_+, \end{split}$		It means to have $m_s = -1$	
on the latter state we will obtain admixtures from possible $\phi_{\textit{nlm};\textit{sm}_{\rm s}}(r,\theta,\phi)$ states.		and	
		will have	









$$""" Call i1 i2 I m1 m2 m1+m2 """$$

Quantum numbers and the Schroeodinger equation in relative and CM coordinates

Summing up, for for the single-particle case, we have the following eigenfunctions

$$\psi_{njm_j;ls} = \sum_{m_lm_s} \langle lm_l sm_s | jm_j \rangle \phi_{nlm_lsm_s}$$

where the coefficients $\langle Im_ism_s|jm_j\rangle$ are the so-called Clebsch-Gordan coefficients. The relevant quantum numbers are n (related to the principal quantum number and the number of nodes of the wave function) and

 $\hat{j}^2 \psi_{njm_j;ls} = \hbar^2 j(j+1)\psi_{njm_j;ls},$

$$j_z \psi_{njm_j;ls} = \hbar m_j \psi_{njm_j;ls},$$

$$I^-\psi_{njm_j;ls} = h^-I(I+1)\psi_{njm_j;ls},$$

 $\hat{s}^2 \psi_{njm_j;ls} = \hbar^2 s(s+1) \psi_{njm_j;ls}$

Quantum numbers and the Schroedinger equation in relative and CM coordinates

For a two-body state where we couple two angular momenta j_1 and j_2 to a final angular momentum J with projection M_J , we can define a similar transformation in terms of the Clebsch-Gordan coeffficients

$$\psi_{(j_1j_2)JM_j} = \sum_{m_{j_1}m_{j_2}} \langle j_1 m_{j_1} j_2 m_{j_2} | JM_j \rangle \psi_{n_1 j_1 m_{j_1}; l_1 s_1} \psi_{n_2 j_2 m_{j_2}; l_2}$$

We will write these functions in a more compact form hereafter, namely,

 $|(j_1j_2)JM_J\rangle = \psi_{(j_1j_2)JM_J},$ $|j_im_{j_i}\rangle = \psi_{n;i_im_{i_1};l_is_i},$

and

We

where we have skipped the explicit reference to I, s and n. The spin of a nucleon is always 1/2 while the value of I can be deduced from the parity of the state. It is thus normal to label a state with

Quantum numbers and the Schroedinger equation in relative and CM coordinates

Our two-body state can thus be written as

$$|(j_1 j_2) J M_J \rangle = \sum_{m_{j_1} m_{j_2}} \langle j_1 m_{j_1} j_2 m_{j_2} | J M_J \rangle | j_1 m_{j_1} \rangle | j_2 m_{j_2} \rangle$$

Due to the coupling order of the Clebsch-Gordan coefficient it reads as j_1 coupled to j_2 to yield a final angular momentum J. If we invert the order of coupling we would have

$$|(j_2j_1)JM_J\rangle = \sum_{m_{j_1}m_{j_2}} \langle j_2m_{j_2}j_1m_{j_1}|JM_J\rangle |j_1m_{j_1}\rangle |j_2m_{j_2}\rangle$$

and due to the symmetry properties of the Clebsch-Gordan coefficient we have

$$|(j_{2j_{1}})JM_{J}\rangle = (-1)^{j_{1}+j_{2}-J} \sum_{m_{j_{1}}m_{j_{2}}} \langle j_{1}m_{j_{1}}j_{2}m_{j_{2}}|JM_{J}\rangle |j_{1}m_{j_{1}}\rangle |j_{2}m_{j_{2}}\rangle = (-1)^{j}$$

Quantum numbers

have thus the coupled basis
$$|(j_1j_2)JM_J\rangle = \sum_{m_{j_1}m_{j_2}} \langle j_1m_{j_1}j_2m_{j_2}|JM_J\rangle |j_1m_{j_1}\rangle |j_2m_{j_2}\rangle.$$

and the uncoupled basis

 $|j_1 m_{j_1}\rangle |j_2 m_{j_2}\rangle.$

The latter can easily be generalized to many single-particle states whereas the first needs specific coupling coefficients and definitions of coupling orders. The *m*-scheme basis is easy to implement numerically and is used in most standard shell-model codes. Our coupled basis obeys also the following relations

 $\hat{J}^2|(j_1j_2)JM_J\rangle = \hbar^2 J(J+1)|(j_1j_2)JM_J\rangle$

$$\hat{J}_{z}|(j_{1}j_{2})JM_{J}\rangle = \hbar M_{J}|(j_{1}j_{2})JM_{J}\rangle$$

Components of the force and isospin

The nuclear forces are almost charge independent. If we assume they are, we can introduce a new quantum number which is conserved. For nucleons only, that is a proton and neutron, we can limit ourselves to two possible values which allow us to distinguish between the two particles. If we assign an isospin value of $\tau = 1/2$ for protons and neutrons (they belong to an isospin doublet, in the same way as we discussed the spin 1/2 multiplet), we can define the neutron to have isospin projection $\tau_z = +1/2$ and a proton to have $\tau_z = -1/2$. These assignements are the standard choices in low-energy nuclear physics.

Isospin

This leads to the introduction of an additional quantum number called isospin. We can define a single-nucleon state function in terms of the quantum numbers n, j, m_j , l, s, τ and τ_z . Using our definitions in terms of an uncoupled basis, we had

$$m_{njm_j;ls} = \sum_{m_lm_s} \langle lm_l sm_s | jm_j
angle \phi_{nlm_lsm_s},$$

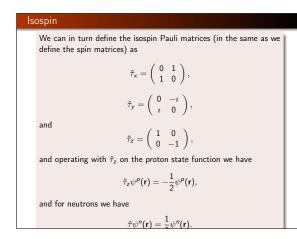
which we can now extend to

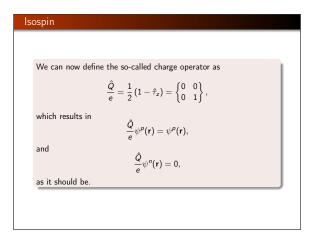
$$\psi_{njm_j;ls}\xi_{\tau\tau_z} = \sum_{m_lm_s} \langle lm_l sm_s | jm_j \rangle \phi_{nlm_lsm_s}\xi_{\tau\tau_z},$$

with the isospin spinors defined as

$$\xi_{\tau=1/2\tau_z=+1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and





Isospin The total isospin is defined as $\hat{T} = \sum_{i=1}^{A} \hat{\tau}_{i},$ and its corresponding isospin projection as $\hat{T}_{z} = \sum_{i=1}^{A} \hat{\tau}_{z_{i}},$ with eigenvalues T(T + 1) for \hat{T} and 1/2(N - Z) for \hat{T}_{z} , where N is the number of neutrons and Z the number of protons. If charge is conserved, the Hamiltonian \hat{H} commutes with \hat{T}_{z} and all members of a given isospin multiplet (that is the same value of T) have the same energy and there is no T_{z} dependence and we

say that \hat{H} is a scalar in isospin space.

Angular momentum algebra, Examples

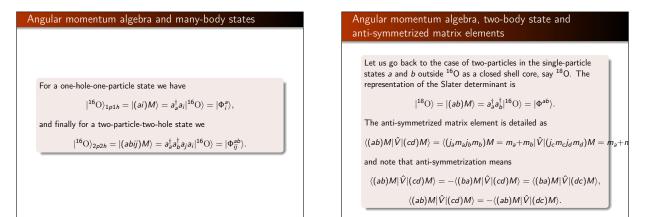
We have till now seen the following definitions of a two-body matrix elements with quantum numbers $p = j_p m_p$ etc we have a two-body state defined as

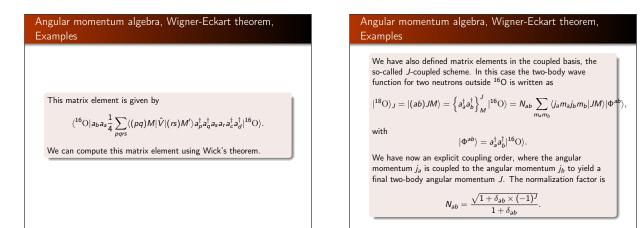
$$(pq)M\rangle = a_p^{\dagger}a_q^{\dagger}|\Phi_0\rangle,$$

where $|\Phi_0\rangle$ is a chosen reference state, say for example the Slater determinant which approximates ^{16}O with the 0s and the 0p shells being filled, and $M=m_p+m_q$. Recall that we label single-particle states above the Fermi level as *abcd* ... and states below the Fermi level for *ijkl* In case of two-particles in the single-particle states a and b outside ^{16}O as a closed shell core, say ^{18}O , we would write the representation of the Slater determinant as

⁸O
$$\rangle = |(ab)M\rangle = a_a^{\dagger}a_b^{\dagger}|^{16}O\rangle = |\Phi^{ab}\rangle.$$

In case of two-particles removed from say $^{16}{\rm O},$ for example two neutrons in the single-particle states i and j, we would write this as





Angular momentum algebra

We note that, using the anti-commuting properties of the creation operators, we obtain

$$N_{ab}\sum_{m_am_b}\langle j_am_aj_bm_b|JM
angle|\Phi^{ab}
angle=-N_{ab}\sum_{m_am_b}\langle j_am_aj_bm_b|JM
angle|\Phi^{ba}
angle.$$

Furthermore, using the property of the Clebsch-Gordan coefficient

$$\langle j_a m_a j_b m_b | JM \rangle = (-1)^{j_a+j_b-J} \langle j_b m_b j_a m_a | JM \rangle$$

which can be used to show that

$$|(j_b j_a) JM\rangle = \left\{a_b^{\dagger} a_a^{\dagger}\right\}_M^J |^{16} O\rangle = N_{ab} \sum_{m_a m_b} \langle j_b m_b j_a m_a | JM\rangle |\Phi^{ba}\rangle,$$

is equal to

 $|(j_b j_a)JM\rangle = (-1)^{j_a+j_b-J+1}|(j_a j_b)JM\rangle.$

Angular momentum algebra, Wigner-Eckart theorem, Examples

The implementation of the Pauli principle looks different in the J-scheme compared with the m-scheme. In the latter, no two fermions or more can have the same set of quantum numbers. In the J-scheme, when we write a state with the shorthand

$$^{18}O\rangle_J = |(ab)JM\rangle,$$

we do refer to the angular momenta only. This means that another way of writing the last state is $% \left({{{\left({{{{\bf{n}}_{\rm{s}}}} \right)}_{\rm{s}}}} \right)$

$$|^{18}O\rangle_J = |(j_a j_b)JM\rangle.$$

We will use this notation throughout when we refer to a two-body state in *J*-scheme. The Kronecker δ function in the normalization factor refers thus to the values of j_a and j_b . If two identical particles are in a state with the same *j*-value, then only even values of the total angular momentum apply. In the notation below, when we label a state as i_a it will actually represent all quantum numbers.

Angular momentum algebra, two-body matrix element

The two-body matrix element is a scalar and since it obeys
rotational symmetry, it is diagonal in *J*, meaning that the
corresponding matrix element in *J*-scheme is
$$\langle (j_{a}j_{b})JM|\hat{V}|(j_{c}j_{d})JM\rangle = N_{ab}N_{cd}\sum_{\substack{m_{a}m_{b}m_{c}m_{d}}}\langle j_{a}m_{a}j_{b}m_{b}|JM\rangle$$

 $\times \langle j_c m_c j_d m_d | JM \rangle \langle (j_a m_a j_b m_b) M | \hat{V} | (j_c m_c j_d m_d) M \rangle,$

and note that of the four *m*-values in the above sum, only three are independent due to the constraint $m_a + m_b = M = m_c + m_d$.

Since

$$\begin{split} |(j_{b}j_{a})JM\rangle &= (-1)^{j_{a}+j_{b}-J+1}|(j_{a}j_{b})JM\rangle, \\ \text{the anti-symmetrized matrix elements need now to obey the following relations} \\ \langle (j_{a}j_{b})JM|\hat{V}|(j_{c}j_{d})JM\rangle &= (-1)^{j_{a}+j_{b}-J+1}\langle (j_{b}j_{a})JM|\hat{V}|(j_{c}j_{d})JM\rangle, \\ \langle (j_{a}j_{b})JM|\hat{V}|(j_{c}j_{d})JM\rangle &= (-1)^{j_{c}+j_{d}-J+1}\langle (j_{a}j_{b})JM|\hat{V}|(j_{d}j_{c})JM\rangle, \\ \langle (j_{a}j_{b})JM|\hat{V}|(j_{c}j_{d})JM\rangle &= (-1)^{j_{a}+j_{b}+j_{c}+j_{d}}\langle (j_{b}j_{a})JM|\hat{V}|(j_{d}j_{c})JM\rangle, \\ \langle (j_{a}j_{b})JM|\hat{V}|(j_{c}j_{d})JM\rangle &= (-1)^{j_{b}+j_{b}+j_{c}+j_{d}}\langle (j_{b}j_{a})JM|\hat{V}|(j_{d}j_{c})JM\rangle = \langle (j_{b}j_{a}), \\ \text{where the last relations follows from the fact that J is an integer and 2J is always an even number. } \end{split}$$

Angular momentum algebra, two-body matrix element
Using the orthogonality properties of the Clebsch-Gordan
coefficients,

$$\sum_{m_a m_b} \langle j_a m_a j_b m_b | JM \rangle \langle j_a m_a j_b m_b | J'M' \rangle = \delta_{JJ'} \delta_{MM'},$$
and

$$\sum_{JM} \langle j_a m_a j_b m_b | JM \rangle \langle j_a m'_a j_b m'_b | JM \rangle = \delta_{m_a m'_b} \delta_{m_b m'_b},$$
we can also express the two-body matrix element in *m*-scheme in
terms of that in *J*-scheme, that is, if we multiply with

$$\sum_{JMJ'M'} \langle j_a m'_a j_b m'_b | JM \rangle \langle j_c m'_c j_d m'_d | J'M' \rangle$$
from left in

$$\langle (j_a j_b) JM | \hat{V} | (j_c j_d) JM \rangle = N_{ab} N_{cd} \sum_{m_a m_b m_c m_d} \langle j_a m_a j_b m_b | JM \rangle \langle j_c m_c j_d m_d | JM \rangle$$

The Hartree-Fock potential

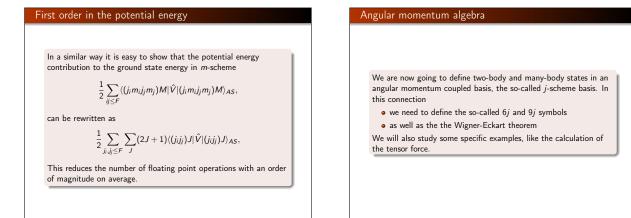
We can now use the above relations to compute the Hartre-Fock energy in j-scheme. In m-scheme we defined the Hartree-Fock energy as

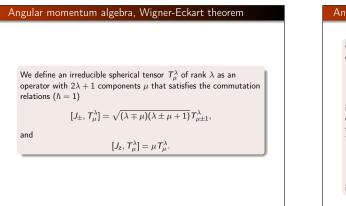
$$_{pq}^{\rm HF} = \delta_{pq} \varepsilon_p + \sum_{i < F} \langle pi | \hat{V} | qi \rangle_{AS},$$

where the single-particle states pqi point to the quantum numbers in *m*-scheme. For a state with for example j = 5/2, this results in six identical values for the above potential. We would obviously like to reduce this to one only by rewriting our equations in *j*-scheme. Our Hartree-Fock basis is orthogonal by definition, meaning that we have

$$\varepsilon_{p}^{\rm HF} = \varepsilon_{p} + \sum_{i \leq F} \langle pi | \hat{V} | pi \rangle_{AS},$$

The Hartree-Fock potential We have $\varepsilon_p^{\rm HF} = \varepsilon_p + \sum_{i \leq F} \langle pi | \hat{V} | pi \rangle_{AS},$ where the single-particle states $p = [n_p, j_p, m_p, t_{z_p}]$. Let us assume that p is a state above the Fermi level. The quantity ε_p could represent the harmonic oscillator single-particle energies. Let $p \to a$. The energies, as we have seen, are independent of m_a and m_i . We sum now over all m_a on both sides of the above equation and divide by $2j_a + 1$, recalling that $\sum_{m_a} = 2j_a + 1$. This results in $\varepsilon_a^{\rm HF} = \varepsilon_a + \frac{1}{2j_a + 1} \sum_{i \leq F} \sum_{m_a} \langle ai | \hat{V} | ai \rangle_{AS},$





Angular momentum algebra, Wigner-Eckart theorem Our angular momentum coupled two-body wave function obeys clearly this definition, namely $|(ab)JM
angle = \left\{a_{a}^{\dagger}a_{b}^{\dagger}
ight\}_{M}^{J}|\Phi_{0}
angle = N_{ab}\sum_{m_{a}m_{b}}\langle j_{a}m_{a}j_{b}m_{b}|JM
angle|\Phi^{ab}
angle,$ is a tensor of rank J with M components. Another well-known example is given by the spherical harmonics (see examples during today's lecture). The product of two irreducible tensor operators $T_{\mu_3}^{\lambda_3} = \sum_{\mu_1\mu_2} \langle \lambda_1 \mu_1 \lambda_2 \mu_2 | \lambda_3 \mu_3 \rangle T_{\mu_1}^{\lambda_1} T_{\mu_2}^{\lambda_2}$ is also a tensor operator of rank λ_3 .

We wish to apply the above definitions to the computations of a matrix element

$$\langle \Phi_M^J | T_\mu^\lambda | \Phi_{M'}^{J'} \rangle,$$

where we have skipped a reference to specific single-particle states. This is the expectation value for two specific states, labelled by angular momenta J' and J. These states form an orthonormal basis. Using the properties of the Clebsch-Gordan coefficients we can write

$$T^\lambda_\mu |\Phi^{J'}_{M'}
angle = \sum_{J''M''} \langle \lambda \mu J'M' |J''M''
angle |\Psi^{J''}_{M''}
angle,$$

and assuming that states with different $J \mbox{ and } M$ are orthonormal we arrive at

 $\langle \Phi^J_M | T^\lambda_\mu | \Phi^{J'}_{M'} \rangle = \langle \lambda \mu J' M' | JM \rangle \langle \Phi^J_M | \Psi^J_M \rangle.$

Angular momentum algebra, Wigner-Eckart theorem

We need to show that $\langle \Phi^J_M | \Psi^J_M \rangle,$

is independent of *M*. To show that

 $\langle \Phi_M^J | \Psi_M^J \rangle$,

is independent of M, we use the ladder operators for angular momentum.

We have that

$$\langle \Phi_{M+1}^{J} | \Psi_{M+1}^{J} \rangle = ((J-M)(J+M+1))^{-1/2} \langle \hat{J}_{+} \Phi_{M}^{J} | \Psi_{M+1}^{J} \rangle,$$

but this is also equal to

$$\langle \Phi_{M+1}^{J} | \Psi_{M+1}^{J} \rangle = ((J-M)(J+M+1))^{-1/2} \langle \Phi_{M}^{J} | \hat{J}_{-} \Psi_{M+1}^{J} \rangle,$$

meaning that

$$\langle \Phi^J_{M+1} | \Psi^J_{M+1} \rangle = \langle \Phi^J_M | \Psi^J_M \rangle \equiv \langle \Phi^J_M || T^{\lambda} || \Phi^{J'}_{M'} \rangle.$$

The double bars indicate that this expectation value is independent of the projection M.

Angular momentum algebra, Wigner-Eckart theorem

The Wigner-Eckart theorem for an expectation value can then be written as

$$\langle \Phi^{J}_{M} | T^{\lambda}_{\mu} | \Phi^{J'}_{M'} \rangle \equiv \langle \lambda \mu J' M' | J M \rangle \langle \Phi^{J} | | T^{\lambda} | | \Phi^{J'} \rangle.$$

The double bars indicate that this expectation value is independent of the projection M. We can manipulate the Clebsch-Gordan coefficients using the relations

 $\langle \lambda \mu J' M' | J M
angle = (-1)^{\lambda + J' - J} \langle J' M' \lambda \mu | J M
angle$

and

$$J'M'\lambda\mu|JM
angle=(-1)^{J'-M'}rac{\sqrt{2J+1}}{\sqrt{2\lambda+1}}\langle J'M'J-M|\lambda-\mu
angle,$$

 $(\Delta J + \tau \lambda + \Delta J') = (-1)J - M (-J - \lambda - J') + (\Delta J + \tau \lambda + \Delta J')$

together with the so-called 3j symbols. It is then normal to encounter the Wigner-Eckart theorem in the form

Angular momentum algebra, Wigner-Eckart theorem
The 3*j* symbols obey the symmetry relation

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^p \begin{pmatrix} j_a & j_b & j_c \\ m_a & m_b & m_c \end{pmatrix},$$
with $(-1)^p = 1$ when the columns *a*, *b*, *c* are even permutations of the columns 1, 2, 3, $p = j_1 + j_2 + j_3$ when the columns *a*, *b*, *c* are odd permations of the columns 1, 2, 3 and $p = j_1 + j_2 + j_3$ when all the magnetic quantum numbers m_i change sign. Their orthogonality is given by

$$\sum_{jam_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1m_1'}\delta_{m_2m_2'},$$
and

$$\sum_{m_1m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_{3'} \end{pmatrix} = \frac{1}{(2j_3+1)} \delta_{jaj_3'} \delta_{m_3m_3'}.$$

Angular momentum algebra, Wigner-Eckart theorem
For later use, the following special cases for the Clebsch-Gordan
and 3*j* symbols are rather useful

$$\langle JMJ'M'|00\rangle = \frac{(-1)^{J-M}}{\sqrt{2J+1}}\delta_{JJ'}\delta_{MM'}.$$
and

$$\begin{pmatrix} J\&1 & J\\ -M\&0 & M' \end{pmatrix} = (-1)^{J-M}\frac{M}{\sqrt{(2J+1)(J+1)}}\delta_{MM'}.$$

Angular momentum algebra, Wigner-Eckart theorem
Using 3j symbols we rewrote the Wigner-Eckart theorem as

$$\langle \Phi_M^J | T_{\lambda}^{\lambda} | \Phi_{M'}^{J'} \rangle \equiv (-1)^{J-M} \begin{pmatrix} J & \lambda & J' \\ -M & \mu & M' \end{pmatrix} \langle \Phi^J || T^{\lambda} || \Phi^{J'} \rangle.$$

Multiplying from the left with the same 3j symbol and summing
over M, μ, M' we obtain the equivalent relation
 $\langle \Phi^J || T^{\lambda} || \Phi^{J'} \rangle \equiv \sum_{M,\mu,M'} (-1)^{J-M} \begin{pmatrix} J & \lambda & J' \\ -M & \mu & M' \end{pmatrix} \langle \Phi_M^J | T_{\lambda}^{\lambda} | \Phi_{M'}^{J'} \rangle,$
where we used the orthogonality properties of the 3j symbols from
the previous page.

Angular momentum algebra, Wigner-Eckart theorem
Similarly, using

$$\begin{pmatrix} J & 1 & J \\ -M & 0 & M' \end{pmatrix} = (-1)^{J-M} \frac{M}{\sqrt{(2J+1)(J+1)}} \delta_{MM'},$$
we have that

$$\langle \Phi^J || \mathbf{J} || \Phi^J \rangle = \sum_{M,M'} (-1)^{J-M} \begin{pmatrix} J & 1 & J' \\ -M & 0 & M' \end{pmatrix} \langle \Phi^J_M | j_Z | \Phi^{J'}_{M'} \rangle = \sqrt{J(J+1)}$$
With the Pauli spin matrices σ and a state with $J = 1/2$, the
reduced matrix element

$$\langle \frac{1}{2} || \sigma || \frac{1}{2} \rangle = \sqrt{6}.$$
Before we proceed with further examples, we need some other
properties of the Wigner-Eckart theorem plus some additional

The Wigner-Eckart theorem states that the expectation value for an irreducible spherical tensor can be written as

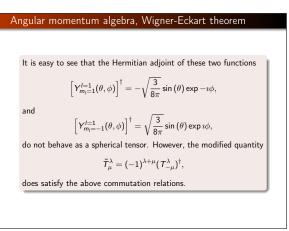
$$\langle \Phi^{J}_{M} | T^{\lambda}_{\mu} | \Phi^{J'}_{M'} \rangle \equiv \langle \lambda \mu J' M' | J M \rangle \langle \Phi^{J} | | T^{\lambda} | | \Phi^{J'} \rangle$$

Since the Clebsch-Gordan coefficients themselves are easy to evaluate, the interesting quantity is the reduced matrix element. Note also that the Clebsch-Gordan coefficients limit via the triangular relation among λ , J and J' the possible non-zero values. From the theorem we see also that

$$\langle \Phi^{J}_{M} | T^{\lambda}_{\mu} | \Phi^{J'}_{M'} \rangle = \frac{\langle \lambda \mu J' M' | JM \rangle \langle}{\langle \lambda \mu_0 J' M'_0 | JM_0 \rangle \langle} \langle \Phi^{J}_{M_0} | T^{\lambda}_{\mu_0} | \Phi^{J'}_{M'_0} \rangle,$$

meaning that if we know the matrix elements for say some $\mu=\mu_0,$ $M'=M'_0$ and $M=M_0$ we can calculate all other.

Angular momentum algebra, Wigner-Eckart theorem	
If we look at the hermitian adjoint of the operator T^{λ}_{μ} , we see via the commutation relations that $(T^{\lambda}_{\mu})^{\dagger}$ is not an irreducible tensor, that is	
$[J_{\pm},(T_{\mu}^{\lambda})^{\dagger}]=-\sqrt{(\lambda\pm\mu)(\lambda\mp\mu+1)}(\mathcal{T}_{\mu\mp1}^{\lambda})^{\dagger},$	
and	
$[J_z,(T^\lambda_\mu)^\dagger]=-\mu(T^\lambda_\mu)^\dagger.$	
The hermitian adjoint $(T^{\lambda}_{\mu})^{\dagger}$ is not an irreducible tensor. As an example, consider the spherical harmonics for $l = 1$ and $m_l = \pm 1$. These functions are	
$Y_{m_l=1}^{l=1}(heta,\phi)=-\sqrt{rac{3}{8\pi}}\sin{(heta)}\exp{\imath\phi},$	
and $Y_{m_l=-1}^{l=1}(heta,\phi)=\sqrt{rac{3}{8\pi}}\sin{(heta)}\exp{-\imath\phi},$	



Angular momentum algebra, Wigner-Eckart theorem

With the modified quantity

angular momenta relations.

$$ilde{T}^{\lambda}_{\mu} = (-1)^{\lambda+\mu} (T^{\lambda}_{-\mu})^{\dagger},$$

we can then define the expectation value

$$\langle \Phi_M^J | T_\mu^\lambda | \Phi_{M'}^{J'} \rangle^\dagger = \langle \lambda \mu J' M' | J M \rangle \langle \Phi^J | | T^\lambda | | \Phi^{J'} \rangle^*,$$

since the Clebsch-Gordan coefficients are real. The rhs is equivalent with

$$\langle \lambda \mu J' M' | J M \rangle \langle \Phi^J | | T^\lambda | | \Phi^{J'} \rangle^* = \langle \Phi^{J'}_{M'} | (T^\lambda_\mu)^\dagger | \Phi^J_M \rangle,$$

which is equal to

 $\langle \Phi_{M'}^{J'} | (\mathcal{T}_{\mu}^{\lambda})^{\dagger} | \Phi_{M}^{J} \rangle = (-1)^{-\lambda+\mu} \langle \lambda - \mu J M | J' M' \rangle \langle \Phi^{J'} | | \tilde{\mathcal{T}}^{\lambda} | | \Phi^{J} \rangle.$

Angular momentum algebra, Wigner-Eckart theorem

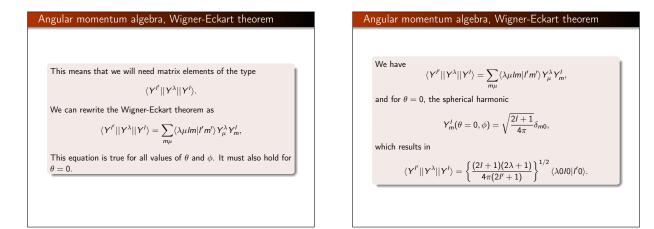
Let us now apply the theorem to some selected expectation values. In several of the expectation values we will meet when evaluating explicit matrix elements, we will have to deal with expectation values involving spherical harmonics. A general central interaction can be expanded in a complete set of functions like the Legendre polynomials, that is, we have an interaction, with $r_{ij} = |r_i - r_j|$,

$$\mathbf{v}(r_{ij}) = \sum_{\nu=0}^{\infty} \mathbf{v}_{\nu}(r_{ij}) P_{\nu}(\cos{(\theta_{ij})}),$$

with P_{ν} being a Legendre polynomials

$$\mathcal{P}_
u(\cos{(heta_{ij})} = \sum_\mu rac{4\pi}{2\mu+1} Y^{
u*}_\mu(\Omega_i) Y^
u_\mu(\Omega_j).$$

We will come back later to how we split the above into a contribution that involves only one of the coordinates.



Till now we have mainly been concerned with the coupling of two angular momenta j_a and j_b to a final angular momentum J. If we wish to describe a three-body state with a final angular momentum J, we need to couple three angular momenta, say the two momenta j_a, j_b to a third one j_c . The coupling order is important and leads to a less trivial implementation of the Pauli principle. With three angular momenta there are obviously 3! ways by which we can combine the angular momenta. In *m*-scheme a three-body Slater determinant is represented as (say for the case of ¹⁹O, three neutrons outside the core of ^{16}O),

$$|^{19}\mathrm{O}\rangle = |(abc)M\rangle = a_a^{\dagger}a_b^{\dagger}a_c^{\dagger}|^{16}\mathrm{O}\rangle = |\Phi^{abc}\rangle.$$

The Pauli principle is automagically implemented via the anti-commutation relations.

Angular momentum algebra, Wigner-Eckart theorem

However, when we deal the same state in an angular momentum coupled basis, we need to be a little bit more careful. We can namely couple the states as follows

$$|([j_a \rightarrow j_b]J_{ab} \rightarrow j_c)J\rangle = \sum_{m_a m_b m_c} \langle j_a m_a j_b m_b | J_{ab} M_{ab} \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} j_c m_c | JM \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} \rangle \langle J_{ab} M_{ab} \rangle \langle J_{ab} M_{ab} \rangle \langle J_{ab} M_{ab} \rangle | j_a h_b \rangle \langle J_{ab} M_{ab} \rangle \langle J_{ab$$

that is, we couple first j_a to j_b to yield an intermediate angular momentum J_{ab} , then to j_c yielding the final angular momentum J.

Angular momentum algebra, Wigner-Eckart theorem

Now, nothing hinders us from recoupling this state by coupling j_b to j_c , yielding an intermediate angular momentum J_{bc} and then couple this angular momentum to j_a , resulting in the final angular momentum J'.

That is, we can have

$$|(j_a \to [j_b \to j_c]J_{bc})J\rangle = \sum_{m'_am'_bm'_c} \langle j_b m'_b j_c m'_c | J_{bc} M_{bc} \rangle \langle j_a m'_a J_{bc} M_{bc} | J' M' \rangle |\Phi|$$

We will always assume that we work with orthornormal states, this means that when we compute the overlap betweem these two possible ways of coupling angular momenta, we get

$$\begin{array}{l} \langle (j_{a} \rightarrow [j_{b} \rightarrow j_{c}] J_{bc}) J' M' | ([j_{a} \rightarrow j_{b}] J_{ab} \rightarrow j_{c}) JM \rangle = \delta_{JJ'} \delta_{MM'} \sum_{\substack{m_{a}m_{b}m_{c} \\ m_{c} \\ (2) \\ \times \langle j_{b}m_{b}j_{c}m_{c}| J_{bc} M_{bc} \rangle \\ (3) \end{array}$$

Angular momentum algebra, Wigner-Eckart theorem

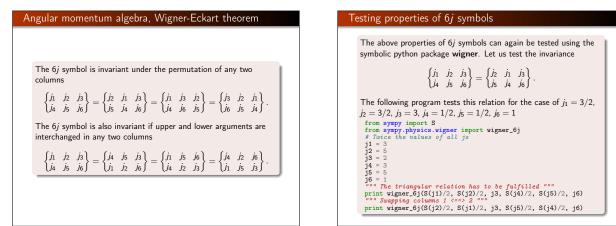
We use then the latter equation to define the so-called 6*j*-symbols

$$\begin{aligned} \langle (j_a \to [j_b \to j_c] J_{bc}) J' M' | ([j_a \to j_b] J_{ab} \to j_c) J M \rangle &= \delta_{JJ'} \delta_{MM'} \sum_{\substack{m_a m_b m_c \\ \times \langle j_b m_b j_c m_c | J_{bc} M \\ = (-1)^{j_a + j_b + j_c + J} \sqrt{\langle 2 \rangle} \end{aligned}$$

(ja

where the symbol in curly brackets is the 6*j* symbol. A specific coupling order has to be respected in the symbol, that is, the so-called triangular relations between three angular momenta needs to be respected, that is

$$\left\{\begin{array}{cc} x & x & x \\ & & \end{array}\right\} \left\{\begin{array}{cc} & x \\ x & x \end{array}\right\} \left\{\begin{array}{cc} x \\ x & x \end{array}\right\} \left\{\begin{array}{cc} x \\ x & x \end{array}\right\} \left\{\begin{array}{cc} x \\ x & x \end{array}\right\}$$



Angular momentum algebra, Wigner-Eckart theorem
The 6j symbols satisfy this orthogonality relation

$$\sum_{j_3} (2j_3+1) \begin{cases} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{cases} \begin{cases} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6' \end{cases} = \frac{\delta_{j_6j_6'}}{2j_6+1} \{j_1, j_5, j_6\} \{j_4, j_2, j_6\}.$$
The symbol $\{j_1j_2j_3\}$ (called the triangular delta) is equal to one if
the triad $(j_1j_2j_3)$ satisfies the triangular conditions and zero
otherwise. A useful value is given when say one of the angular
momenta are zero, say $J_{bc} = 0$, then we have

$$\begin{cases} j_a & j_b & J_{ab} \\ j_c & J & 0 \end{cases} = \frac{(-1)^{j_a+j_b+J_{ab}}\delta_{Jj_b}\delta_{j_cj_b}}{\sqrt{(2j_b+1)(2j_b+1)}}$$

With the 6j symbol defined, we can go back and and rewrite the overlap between the two ways of recoupling angular momenta in terms of the 6j symbol. That is, we can have

$$|(j_{a} \rightarrow [j_{b} \rightarrow j_{c}]J_{bc})JM\rangle = \sum_{J_{ab}} (-1)^{j_{a}+j_{b}+j_{c}+J} \sqrt{(2J_{ab}+1)(2J_{bc}+1)} \begin{cases} j \\ j \end{cases}$$

Can you find the inverse relation? These relations can in turn be used to write out the fully anti-symmetrized three-body wave function in a J-scheme coupled basis. If you opt then for a specific coupling order, say $|(f_{Ja} \rightarrow j_{B}|J_{ab} \rightarrow j_{c})JM\rangle$, you need to express this representation in terms of the other coupling possibilities.

Angular momentum algebra, Wigner-Eckart theorem

Note that the two-body intermediate state is assumed to be antisymmetric but not normalized, that is, the state which involves the quantum numbers j_a and j_b . Assume that the intermediate two-body state is antisymmetric. With this coupling order, we can rewrite (in a schematic way) the general three-particle Slater determinant as

$$\Phi(a, b, c) = \mathcal{A}|([j_a \to j_b]J_{ab} \to j_c)J\rangle,$$

with an implicit sum over J_{ab} . The antisymmetrization operator ${\mathcal A}$ is used here to indicate that we need to antisymmetrize the state. **Challenge**: Use the definition of the 6j symbol and find an explicit expression for the above three-body state using the coupling order $|([j_a \rightarrow j_b]J_{ab} \rightarrow j_c)J\rangle.$

Angular momentum algebra, Wigner-Eckart theorem

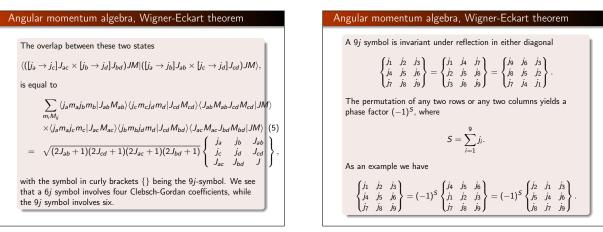
We can also coupled together four angular momenta. Consider two four-body states, with single-particle angular momenta j_a , j_b , j_c and j_d we can have a state with final J

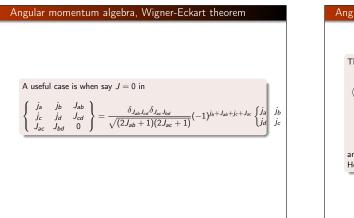
$$\Phi(a, b, c, d)\rangle_1 = |([j_a \rightarrow j_b]J_{ab} \times [j_c \rightarrow j_d]J_{cd})JM\rangle,$$

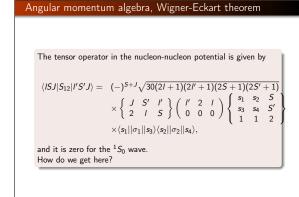
where we read the coupling order as j_a couples with j_b to given and intermediate angular momentum J_{ab} . Moreover, j_c couples with j_d to given and intermediate angular momentum J_{cd} . The two intermediate angular momenta J_{ab} and J_{cd} are in turn coupled to a final J. These operations involved three Clebsch-Gordan coefficients.

Alternatively, we could couple in the following order

 $|\Phi(a, b, c, d)\rangle_2 = |([j_a \rightarrow j_c]J_{ac} \times [j_b \rightarrow j_d]J_{bd})JM\rangle,$







Angular momentum algebra, Wigner-Eckart theoremAngulaTo derive the expectation value of the nuclear tensor force, we
recall that the product of two irreducible tensor operators isStart
$$W'_{m_r} = \sum_{m_p m_q} \langle pm_p qm_q | rm_r \rangle T^p_{m_p} U^q_{m_q}$$
,
and using the orthogonality properties of the Clebsch-Gordan
coefficients we can rewrite the above aswe a
 $T^p_{m_p} U^q_{m_q} = \sum_{m_p m_q} \langle pm_p qm_q | rm_r \rangle W^r_{m_r}$.Assume now that the operators T and U act on different parts of
say a wave function. The operator T could act on the spatial part
only while the operator U acts only on the spin part. This means
also that these operators commute. The reduced matrix element of
this operator is thus, using the Wigner-Eckart theorem, $\langle (j_a j_p) J | |W'| | (j_c j_d) J' \rangle \equiv \sum_{m_r} (-1)^{J-M} \begin{pmatrix} J & r & J' \\ m_r & m_r & M' \end{pmatrix}$

Angular momentum algebra, Wigner-Eckart theorem
Starting with

$$\langle (j_{ajb})J||W^{r}||(j_{c}j_{d})J'\rangle \equiv \sum_{M,m_{r},M'} (-1)^{J-M} \begin{pmatrix} J & r & J' \\ -M & m_{r} & M' \end{pmatrix} \\
\times \langle (j_{a}j_{b}JM| \begin{bmatrix} T_{m_{p}}^{p}U_{m_{q}}^{q} \end{bmatrix}_{m_{r}}^{r} |(j_{c}j_{d})J'M'\rangle,$$
we assume now that T acts only on j_{a} and j_{c} and that U acts only on j_{b} and j_{d} . The matrix element
 $\langle (j_{ajb}JM| \begin{bmatrix} T_{m_{p}}^{p}U_{m_{q}}^{q} \end{bmatrix}_{m_{r}}^{r} |(j_{c}j_{d})J'M'\rangle$ can be written out, when we insert a complete set of states $|j_{i}m_{i}j_{j}m_{j}\rangle\langle j_{i}m_{i}j_{j}m_{j}|$ between T and U as
 $\langle (j_{ajb}JM| \begin{bmatrix} T_{m_{p}}^{p}U_{m_{q}}^{q} \end{bmatrix}_{m_{r}}^{r} |(j_{c}j_{d})J'M'\rangle = \sum_{m_{i}} \langle pm_{p}qm_{q}|rm_{r}\rangle\langle j_{a}m_{a}j_{b}m_{b}|JM\rangle\langle j$
 $\times \langle (j_{a}m_{a}j_{b}m_{b}| \begin{bmatrix} T_{m_{p}}^{p} \end{bmatrix}_{m_{r}}^{r} |(j_{c}m_{c}j_{b}m_{b})\rangle\langle (j_{c}m_{c}j_{b}m_{b}| \begin{bmatrix} U_{m_{q}}^{q} \end{bmatrix}_{m_{r}}^{r} |(j_{c}m_{c}j_{d}m_{d})\rangle.$
The complete set of states that was inserted between T and U

Angular momentum algebra, Wigner-Eckart theorem
Combining the last two equations from the previous slide and and applying the Wigner-Eckart theorem, we arrive at (rearranging phase factors)

$$\langle (j_{ajb})J||W^{r}||(j_{cjd})J^{r}\rangle = \sqrt{(2J+1)(2r+1)(2J^{r}+1)} \sum_{m,M,M^{r}} \begin{pmatrix} J \\ -M \\ m_{r} \end{pmatrix} \begin{pmatrix} r \\ m_{r} \end{pmatrix} \begin{pmatrix} r \\ m_{r} \end{pmatrix} \begin{pmatrix} j_{a} & j_{b} & J \\ j_{c} & j_{a} & J \end{pmatrix} = \sqrt{(2J+1)(2r+1)(2J^{r}+1)} \sum_{m,M,M^{r}} \begin{pmatrix} J \\ -M \\ m_{r} \end{pmatrix} \begin{pmatrix} r \\ m_{r} \end{pmatrix} \begin{pmatrix} j_{a} & j_{b} & J \\ j_{c} & j_{a} & J \end{pmatrix} = \frac{\delta_{JJ^{r}}\delta_{pq}}{\sqrt{(2J+1)(2J+1)}} (-1)^{j_{b}+j_{c}+2J} \left\{ j_{a} & j_{b} & J \\ j_{d} & j_{c} & p \end{pmatrix} \\ \times \begin{pmatrix} j_{a} & j_{b} & J \\ m_{a} & -m_{c} & -m_{p} \end{pmatrix} \begin{pmatrix} j_{b} & j_{d} & q \\ m_{b} & -m_{d} & -m_{q} \end{pmatrix} \langle j_{a}||T^{p}||j_{c}\rangle \times \langle j_{b}||U^{q}||j_{d}\rangle \\ \text{which can be rewritten in terms of a 9j symbol as} \\ \langle (j_{ajb})J||W^{r}||(j_{cjd})J^{r}\rangle = \sqrt{(2J+1)(2r+1)(2J^{r}+1)} \langle j_{a}||T^{p}||j_{c}\rangle \langle j_{b}||U^{q}||j_{d}\rangle \\ \text{which can be rewritten in terms of a 9j symbol as} \end{cases}$$

Angular momentum

Ingular momentum algebra, Wigner-Eckart theorem
We need that the coordinate vector **r** can be written in terms of spherical components as

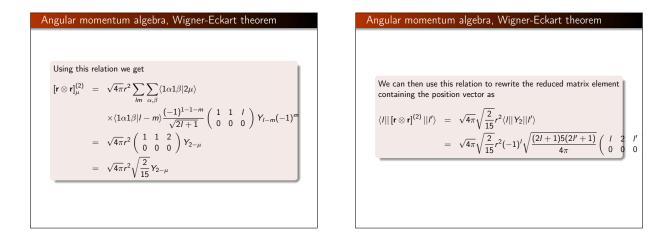
$$\mathbf{r}_{\alpha} = r\sqrt{\frac{4\pi}{3}}\mathbf{Y}_{1\alpha}$$

Using this expression we get
 $[\mathbf{r} \otimes \mathbf{r}]_{\mu}^{(2)} = \frac{4\pi}{3}r^{2}\sum_{\alpha,\beta}\langle 1\alpha 1\beta | 2\mu \rangle \mathbf{Y}_{1\alpha}\mathbf{Y}_{1\beta}$
Angular momentum algebra, Wigner-Eckart
The product of two spherical harmonics can be written
 $\mathbf{Y}_{l_{1}m_{1}}\mathbf{Y}_{l_{2}m_{2}} = \sum_{l_{m}}\sqrt{\frac{(2l_{1}+1)(2l_{2}+1)(2l_{1}+1)}{4\pi}} \begin{pmatrix} r_{1} \\ r_{2} \end{pmatrix} \mathbf{Y}_{l_{1}-m_{1}}(-1)^{m}$

Angular momentum algebra, Wigner-Eckart theorem
The product of two spherical harmonics can be written as

$$Y_{l_1m_1}Y_{l_2m_2} = \sum_{lm} \sqrt{\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}} \begin{pmatrix} h_1 & h_2 & l \\ m_1 & m_2 & m \end{pmatrix}$$

$$\times \begin{pmatrix} h_1 & h_2 & l \\ 0 & 0 & 0 \end{pmatrix} Y_{l-m}(-1)^m.$$



Angular momentum algebra, Wigner-Eckart theorem
Using the reduced matrix element of the spin operators defined as

$$\langle S|| [\sigma_1 \otimes \sigma_2]^{(2)} ||S'\rangle = \sqrt{(2S+1)(2S'+1)5} \begin{cases} s_1 & s_2 & S \\ s_3 & s_4 & S' \\ 1 & 1 & 2 \end{cases}$$

$$\times \langle s_1||\sigma_1||s_3\rangle\langle s_2||\sigma_2||s_4\rangle$$
and inserting these expressions for the two reduced matrix elements
we get

$$\langle ISJ|V|I'S'J\rangle = (-1)^{S+J}\sqrt{30(2I+1)(2I'+1)(2S+1)(2S'+1)}$$

$$\times \begin{cases} I & S & J \\ I' & S & 2 \end{cases} \begin{pmatrix} I & 2 & I' \\ 0 & 0 & 0 \end{pmatrix} \begin{cases} s_1 & s_2 & S \\ s_3 & s_4 & S' \\ 1 & 1 & 2 \end{cases}$$

$$\times \langle s_1||\sigma_1||s_3\rangle\langle s_2||\sigma_2||s_4\rangle.$$

Normally, we start we a nucleon-nucleon interaction fitted to reproduce scattering data. It is common then to represent this

Angular momentum algebra, Wigner-Eckart theorem

reproduce scattering data. It is common then to represent this interaction in terms relative momenta k, the center-of-mass momentum K and various partial wave quantum numbers like the spin S, the total relative angular momentum \mathcal{J} , isospin T and relative orbital momentum I and finally the corresponding center-of-mass L. We can then write the free interaction matrix V as

$\langle kKILJST|\hat{V}|k'KI'LJS'T\rangle.$

Transformations from the relative and center-of-mass motion system to the lab system will be discussed below.

Angular momentum algebra, Wigner-Eckart theorem To obtain a V-matrix in a h.o. basis, we need the transformation $\langle nNIL \mathcal{J}ST | \hat{V} | n'N'I'L' \mathcal{J}S'T \rangle$, with *n* and *N* the principal quantum numbers of the relative and center-of-mass motion, respectively. $|nINL\mathcal{J}ST \rangle = \int k^2 K^2 dk dK R_{nl}(\sqrt{2}\alpha k) R_{NL}(\sqrt{1/2}\alpha K) |klKL\mathcal{J}ST \rangle$. The parameter α is the chosen oscillator length.

Angular momentum algebra, Wigner-Eckart theorem

The most commonly employed sp basis is the harmonic oscillator, which in turn means that a two-particle wave function with total angular momentum J and isospin T can be expressed as

$$|(n_{a}l_{a}j_{a})(n_{b}l_{b}j_{b})JT\rangle = \frac{1}{\sqrt{(1+\delta_{12})}} \sum_{\lambda S \mathcal{J}} \sum_{nNlL} F \times \langle ab|\lambda S J \rangle \\ \times (-1)^{\lambda+\mathcal{J}-L-S} \hat{\lambda} \left\{ \begin{array}{cc} L & I & \lambda \\ S & J & \mathcal{J} \end{array} \right\} \\ \times \langle n|NL|n_{a}l_{a}n_{b}l_{b} \rangle |n|NL|\mathcal{J}ST \rangle.$$

where the term $\langle nINL|n_a I_a n_b I_b \rangle$ is the so-called Moshinsky-Talmi transformation coefficient (see chapter 18 of Alex Brown's notes).

