Electromagnetic transitions and β-decay

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Spring 2016

Reminder on the Wigner-Eckart theorem

In our angular momentum lectures on the Wigner-Eckart theorem we developed two equations. One for the general expectation value that depends also on the magnetic quantum numbers

$$\langle \Phi^J_M | T^\lambda_\mu | \Phi^{J'}_{M'} \rangle \equiv (-1)^{J-M} \begin{pmatrix} J & \lambda & J' \\ -M & \mu & M' \end{pmatrix} \langle \Phi^J || T^\lambda || \Phi^{J'} \rangle,$$

and one for the reduced matrix elements in terms of

$$\langle \Phi^J || T^\lambda || \Phi^{J'} \rangle \equiv \sum_{M,\mu,M'} (-1)^{J-M} \begin{pmatrix} J & \lambda & J' \\ -M & \mu & M' \end{pmatrix} \langle \Phi^J_M | T^\lambda_\mu | \Phi^{J'}_{M'} \rangle.$$

Why the Wigner-Eckart theorem?

Unless we have observables which depend on the magnetic quantum numbers, the degeneracy given by these quantum numbers is not seen experimentally. The typical situation when we perform shell-model calculations is that the results depend on the magnetic quantum numbers. The reason for this is that it is easy to implement the Pauli principle for many particles when we work in what we dubbed for m-scheme.

A resulting state in a shell-model calculations will thus depend on the total value of M defined as

$$M = \sum_{i=1}^{A} m_{j_i}.$$

A shell model many-body state is given by a linear combination of Slater determinants $|\Phi_i\rangle$. That is, for some conserved quantum numbers λ we have

$$|Psi_\lambda\rangle = \sum_i C_i |\Phi_i\rangle,$$

Why the Wigner-Eckart theorem, representing a shell-model state

In second quantization, our ansatz for a state like the ground state is

$$|\Phi_0\rangle = \left(\prod_{i \le F} \hat{a}^\dagger_i \right) |0\rangle,$$

where the index i defines different single-particle states up to the Fermi level. We have assumed that we have N fermions. A given one-particle-one-hole ($1p1h$) state can be written as

$$|\Phi^a_i\rangle = \hat{a}^\dagger_a \hat{a}_i |\Phi_0\rangle,$$

while a $2p2h$ state can be written as

$$|\Phi^{ab}_{ij}\rangle = \hat{a}^\dagger_a \hat{a}^\dagger_b \hat{a}_j \hat{a}_i |\Phi_0\rangle,$$

and a general $NpNh$ state as

$$|\Phi^{abc\dots}_{ijk\dots}\rangle = \hat{a}^\dagger_a \hat{a}^\dagger_b \hat{a}^\dagger_c \dots \hat{a}_k \hat{a}_j \hat{a}_i |\Phi_0\rangle.$$

Representing a shell-model state

A general shell-model many-body state

$$|Psi_\gamma\rangle = \sum_i C_i |\Phi_i\rangle,$$

can be expanded as

$$|\Psi_\gamma\rangle = C_0 |\Phi_0\rangle + \sum_{ai} C^a_i |\Phi^a_i\rangle + \sum_{abij} C^{ab}_{ij} |\Phi^{ab}_{ij}\rangle + \dots .$$

Representing a shell-model state and one-body operators

A one-body operator represented by a spherical tensor of rank λ is given as

$$O^\lambda_\mu = \sum_{pq} \langle p | O^\lambda_\mu | q \rangle a^\dagger_p a_q,$$

meaning that when we compute a transition amplitude

$$\langle \Psi_\delta | O^\lambda_\mu | \Psi_\gamma \rangle = \sum_{ij} C^*_{\delta i} C_{\gamma j} \langle \Phi_i | O^\lambda_\mu | \Phi_j \rangle,$$

we need to compute

$$\langle \Phi_i | O^\lambda_\mu | \Phi_j \rangle.$$

Rewriting the transition amplitude

We want to rewrite

$$\langle\Phi_i|O_\mu^\lambda|\Phi_j\rangle,$$

in terms of the reduced matrix element only. Let us introduce the relevant quantum numbers for the states Φ_i and Φ_j. We include only the relevant ones. We have then in m-scheme

$$\langle\Phi_i|O_\mu^\lambda|\Phi_j\rangle=\sum_{pq}\langle p|O_\mu^\lambda|q\rangle\langle\Phi_M^J|a_p^\dagger a_q|\Phi_{M'}^{J'}\rangle.$$

With a shell-model m-scheme basis it is straightforward to compute these amplitudes. However, as mentioned above, if we wish to related these elements to experiment, we need to use the Wigner-Eckart theorem and express the amplitudes in terms of reduced matrix elements.

Rewriting the transition amplitude, first step

We can rewrite the above transition amplitude using the Wigner-Eckart theorem. Our first step is to rewrite the one-body operator in terms of reduced matrix elements. Since the operator is a spherical tensor we need that the annihilation operator is rewritten as (where q represents j_q, m_q etc)

$$\tilde{a}_q=(-1)^{j_q-m_q}a_{j_q,m_q}.$$

The operator

$$O_\mu^\lambda=\sum_{pq}\langle p|O_\mu^\lambda|q\rangle a_p^\dagger a_q,$$

is rewritten using the Wigner-Eckart theorem as

$$O_\mu^\lambda=\sum_{pq}\langle p||O^\lambda||q\rangle(-1)^{j_p-m_p}\begin{pmatrix} j_p & \lambda & j_q\\ -m_p & \mu & m_q\end{pmatrix}a_p^\dagger a_q.$$

Rewriting the transition amplitude, second step

We have

$$O_\mu^\lambda=\sum_{pq}\langle p||O^\lambda||q\rangle(-1)^{j_p-m_p}\begin{pmatrix} j_p & \lambda & j_q\\ -m_p & \mu & m_q\end{pmatrix}a_p^\dagger a_q.$$

We then single out the sum over m_p and m_q only and define the recoupled one-body part of the operator as

$$\hat{\lambda}^{-1}\left[a_{j_p}^\dagger\tilde{a}_{j_q}\right]_\mu^\lambda=\sum_{m_p,m_q}(-1)^{j_p-m_p}\begin{pmatrix} j_p & \lambda & j_q\\ -m_p & \mu & m_q\end{pmatrix}a_p^\dagger a_q,$$

with $\hat{\lambda}=\sqrt{2\lambda+1}$. This gives the following expression for the one-body operator

$$O_\mu^\lambda=\sum_{j_pj_q}\langle p||O^\lambda||q\rangle\hat{\lambda}^{-1}\left[a_{j_p}^\dagger\tilde{a}_{j_q}\right]_\mu^\lambda.$$

Rewriting the transition amplitude, third step

With

$$O_\mu^\lambda=\sum_{j_pj_q}\langle p||O^\lambda||q\rangle\hat{\lambda}^{-1}\left[a_{j_p}^\dagger\tilde{a}_{j_q}\right]_\mu^\lambda,$$

we can write

$$\langle\Phi_M^J|O_\mu^\lambda|\Phi_{M'}^{J'}\rangle=\sum_{pq}\langle p|O_\mu^\lambda|q\rangle\langle\Phi_M^J|a_p^\dagger a_q|\Phi_{M'}^{J'}\rangle,$$

as

$$\langle\Phi_M^J|O_\mu^\lambda|\Phi_{M'}^{J'}\rangle=\sum_{j_pj_q}\langle p||O^\lambda||q\rangle\langle\Phi_M^J|\hat{\lambda}^{-1}\left[a_{j_p}^\dagger\tilde{a}_{j_q}\right]_\mu^\lambda|\Phi_{M'}^{J'}\rangle.$$

We have suppressed the summation over quantum numbers like n_p, n_q etc.

Rewriting the transition amplitude, final step

Using the Wigner-Eckart theorem

$$\langle\Phi_M^J|O_\mu^\lambda|\Phi_{M'}^{J'}\rangle\equiv(-1)^{J-M}\begin{pmatrix} J & \lambda & J'\\ -M & \mu & M'\end{pmatrix}\langle\Phi^J||O^\lambda||\Phi^{J'}\rangle,$$

we can then define

$$\langle\Phi^J||O^\lambda||\Phi^{J'}\rangle=\hat{\lambda}^{-1}\sum_{j_pj_q}\langle p||O^\lambda||q\rangle\langle\Phi_M^J||\left[a_{j_p}^\dagger\tilde{a}_{j_q}\right]^\lambda||\Phi_{M'}^{J'}\rangle.$$

The quantity to the left in the last equation is normally called the transition amplitude or in case of a decay process, simply the decay amplitude. The quantity $\langle\Phi_M^J|\hat{\lambda}^{-1}\left[a_{j_p}^\dagger\tilde{a}_{j_q}\right]_\mu^\lambda|\Phi_{M'}^{J'}\rangle$ is called the one-body transition density while the corresponding reduced one is simply called the reduced one-body transition density. The transition densities characterize the many-nucleon properties of the initial and final states. They do not carry information about the transition operator beyond its one-body character. Finally, note that in a shell-model calculation it is actually $\langle\Phi_M^J|O^\lambda|\Phi_{M'}^{J'}\rangle$ which

Electromagnetic multipole moments and transitions

The reduced transition probability B is defined in terms of reduced matrix elements of a one-body operator by

$$B(i\rightarrow f)=\frac{\langle J_f||\mathcal{O}(\lambda)||J_i\rangle^2}{(2J_i+1)}.$$

With our definition of the reduced matrix element,

$$\langle J_f||\mathcal{O}(\lambda)||J_i\rangle^2=\langle J_i||\mathcal{O}(\lambda)||J_f\rangle^2,$$

the transition probability B depends upon the direction of the transition by the factor of $(2J_i+1)$. For electromagnetic transitions J_i is that for the higher-energy initial state. But in Coulomb excitation the initial state is usually taken as the ground state, and it is normal to use the notation $B(\uparrow)$ for transitions from the ground state.

The one-body operators $\mathcal{O}(\lambda)$ represent a sum over the operators for the individual nucleon degrees of freedom i

$$\mathcal{O}(\lambda) = \sum_i O(\lambda, i).$$

The electric transition operator is given by

$$O(E\lambda) = r^\lambda \, Y^\lambda_\mu(\hat{r}) \, e_q e,$$

were Y^λ_μ are the spherical harmonics and q stands for proton $q = p$ or neutron $q = n$.

Gamma transitions with $\lambda = 0$ are forbidden because the photon must carry off at least one unit of angular momentum. The e_q are the electric charges for the proton and neutron in units of e. For the free-nucleon charge we would take $e_p = 1$ and $e_n = 0$, for the proton and neutron, respectively. Although the bare operator acts upon the protons, we will keep the general expression in terms of e_q in order to incorporate the **effective charges** for the proton and neutron, which represent the center-of-mass corrections and the average effects of the renormalization from wavefunction admixtures outside the model space.

The magnetic transition operator is given by:

$$O(M\lambda) = \left[\mathbf{l}\frac{2g^l_q}{(\lambda+1)} + \mathbf{s}g^s_q\right]\nabla[r^\lambda Y^\lambda_\mu(\hat{r})]\mu_N$$

$$= \sqrt{\lambda(2\lambda+1)}\left[[Y^{\lambda-1}(\hat{r})\otimes\mathbf{l}]^\lambda_\mu\frac{2g^l_q}{(\lambda+1)} + [Y^{\lambda-1}(\hat{r})\otimes\mathbf{s}]^\lambda_\mu g^s_q\right] r^{\lambda-1}\mu_N,$$

where μ_N is the nuclear magneton,

$$\mu_N = \frac{e\hbar}{2m_pc} = 0.105 \text{ efm},$$

and where m_p is the mass of the proton.

The g-factors g^l_q and g^s_q are the orbital and spin g-factors for the proton and neutron, respectively. The free-nucleon values for the g-factors are $g^l_p = 1$, $g^l_n = 0$, $g^s_p = 5.586$ and $g^s_n = -3.826$. We may use effective values for these g-factors to take into account the truncation of the model space.

The most common types of transitions are $E1$, $E2$ and $M1$. The $E1$ transition operator is given by $\lambda = 1$

$$O(E1) = rY^{(1)}_\mu(\hat{r})e_q e = \sqrt{\frac{3}{4\pi}}\mathbf{r}e_q e.$$

The $E2$ transition operator with $\lambda = 2$

$$O(E2) = r^2 Y^{(2)}_\mu(\hat{r})e_q e,$$

The $M1$ transition operator with $\lambda = 1$ and with

$$Y^0 = 1/\sqrt{4\pi},$$

we have

$$O(M1) = \sqrt{\frac{3}{4\pi}}[\mathbf{l}g^l_q + \mathbf{s}\, g^s_q]\mu_N.$$

Electromagnetic multipole moments and transitions

The selection rules are given by the triangle condition for the angular momenta, $\Delta(J_i, J_f, \lambda)$.
The electromagnetic interaction conserves parity, and the elements of the operators for $E\lambda$ and $M\lambda$ can be classified according to their transformation under parity change

$$\hat{P}\hat{O}\hat{P}^{-1} = \pi_O \hat{O},$$

where we have $\pi_O = (-1)^\lambda$ for Y^λ, $\pi_O = -1$ for the vectors $\mathbf{r}$, ∇ and $\mathbf{p}$, and $\pi_O = +1$ for the pseudo vectors $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ and σ. For a given matrix element we have:

$$\langle \Psi_f | \mathcal{O} | \Psi_i \rangle = \langle \Psi_f | P^{-1} P \mathcal{O} P^{-1} P | \Psi_i \rangle = \pi_i \pi_f \pi_O \langle \Psi_f | \mathcal{O} | \Psi_i \rangle.$$

The matrix element will vanish unless $\pi_i \pi_f \pi_O = +1$.

Electromagnetic multipole moments and transitions

The transitions are divided into two classes, those which do not change parity change $\pi_i \pi_f = +1$ which go by the operators with $\pi_O = +1$:

$$\pi_i \pi_f = +1 \text{ for } M1, E2, M3, E4 \ldots,$$

and the ones which do change parity change $\pi_i \pi_f = -1$ which go by the operators with $\pi_O = -1$:

$$\pi_i \pi_f = -1 \text{ for } E1, M2, E3, M4 \ldots.$$

Electromagnetic multipole moments and transitions

The electromagnetic moment operator can be expressed in terms of the electromagnetic transition operators. By the parity selection rule of the moments are nonzero only for $M1$, $E2$, $M3$, $E4, \ldots$. The most common are:

$$\mu = \sqrt{\frac{4\pi}{3}} \langle J, M = J | \mathcal{O}(M1) | J, M = J \rangle = \sqrt{\frac{4\pi}{3}} \left\{ \begin{array}{ccc} J & 1 & J \\ -J & 0 & J \end{array} \right\} \langle J || \mathcal{O}(M$$

and

$$Q = \sqrt{\frac{16\pi}{5}} \langle J, M = J | \mathcal{O}(E2) | J, M = J \rangle = \sqrt{\frac{16\pi}{5}} \begin{pmatrix} J & 2 & J \\ -J & 0 & J \end{pmatrix} \langle J || \mathcal{O}($$

Electromagnetic multipole moments and transitions

Electromagnetic transitions and moments depend upon the reduced nuclear matrix elements $\langle f || \mathcal{O}(\lambda) || i \rangle$. These can be expressed as a sum over one-body transition densities (OBTD) times single-particle matrix elements

$$\langle f || \mathcal{O}(\lambda) || i \rangle = \sum_{k_\alpha k_\beta} \mathrm{OBTD}(fik_\alpha k_\beta \lambda) \langle k_\alpha || O(\lambda) || k_\beta \rangle,$$

where the OBTD is given by

$$\mathrm{OBTD}(fik_\alpha k_\beta \lambda) = \frac{\langle f || [a^+_{k_\alpha} \otimes \tilde{a}_{k_\beta}]^\lambda || i \rangle}{\sqrt{(2\lambda + 1)}}.$$

The labels i and f are a short-hand notation for the initial and final state quantum numbers $(n\omega_i J_i)$ and $(n\omega_f J_f)$, respectively. Thus the problem is divided into two parts, one involving the nuclear structure dependent one-body transition densities OBTD, and the other involving the reduced single-particle matrix elements (SPME).

Electromagnetic multipole moments and transitions

The SPME for the $E\lambda$ operator is given by

$$\langle k_a || O(E\lambda) || k_b \rangle = (-1)^{j_a + 1/2} \frac{[1 + (-1)^{l_a + \lambda + l_b}]}{2}$$

$$\times \sqrt{\frac{(2j_a + 1)(2\lambda + 1)(2j_b + 1)}{4\pi}} \begin{pmatrix} j_a & \lambda & j_b \\ 1/2 & 0 & -1/2 \end{pmatrix} \langle k_a | r^\lambda | k_b \rangle e_q e.$$

Electromagnetic multipole moments and transitions

The SPME for the spin part of the magnetic operator is

$$\langle k_a || O(M\lambda, s) || k_b \rangle =$$

$$= \sqrt{\lambda(2\lambda + 1)} < j_a || [Y^{\lambda - 1}(\hat{r}) \otimes \mathbf{s}]^\lambda || j_b >< k_a | r^{\lambda - 1} | k_b > g_q^s \mu_N,$$

$$= \sqrt{\lambda(2\lambda + 1)} \sqrt{(2j_a + 1)(2j_b + 1)(2\lambda + 1)} \left\{ \begin{array}{ccc} l_a & 1/2 & j_a \\ l_b & 1/2 & j_b \\ \lambda - 1 & 1 & \lambda \end{array} \right\}$$

$$\times \langle l_a || Y^{\lambda - 1}(\hat{r}) || l_b \rangle \langle || \mathbf{s} || s \rangle \langle k_a | r^{\lambda - 1} | k_b \rangle g_q^s \mu_N,$$

where

$$\langle || \mathbf{s} || s \rangle = \sqrt{3/2}.$$

Electromagnetic multipole moments and transitions

The SPME for the orbital part of the magnetic operator is:

$$\langle k_a||O(M\lambda,l)||k_b\rangle =$$

$$= \frac{\sqrt{\lambda(2\lambda+1)}}{\lambda+1}\langle j_a||[Y^{\lambda-1}(\hat{r})\otimes \mathbf{l}]^{\lambda}||j_b\rangle\langle k_a|r^{\lambda-1}|k_b\rangle g_q^l \mu_N$$

$$= \frac{\sqrt{\lambda(2\lambda+1)}}{\lambda+1}(-1)^{l_a+1/2+j_b+\lambda}\sqrt{(2j_a+1)(2j_b+1)}$$

$$\times \left\{ \begin{array}{ccc} l_a & l_b & \lambda \\ j_b & j_a & 1/2 \end{array} \right\} \langle l_a||[Y^{\lambda-1}(\hat{r})\otimes \mathbf{l}]^{\lambda}||l_b\rangle\langle k_a|r^{\lambda-1}|k_b\rangle g_q^l \mu_N,$$

Electromagnetic multipole moments and transitions

where we have defined

$$\langle l_a||[Y^{\lambda-1}(\hat{r})\otimes \mathbf{l}]^{\lambda}||l_b\rangle = (-1)^{\lambda+l_a+l_b}\sqrt{(2\lambda+1)l_b(l_b+1)(2l_b+1)}$$

$$\times \left\{ \begin{array}{ccc} \lambda-1 & 1 & \lambda \\ l_b & l_a & l_b \end{array} \right\} \langle l_a||Y^{\lambda-1}(\hat{r})||l_b\rangle,$$

with

$$\langle l_a||Y^{\lambda-1}(\hat{r})||l_b\rangle = (-1)^{l_a}\sqrt{\frac{(2l_a+1)(2l_b+1)(2\lambda-1)}{4\pi}}\left(\begin{array}{ccc} l_a & \lambda-1 & l_b \\ 0 & 0 & 0 \end{array} \right)$$

Electromagnetic multipole moments and transitions

For the $M1$ operator the radial matrix element is

$$< k_a|r^0|k_b> = \delta_{n_a,n_b},$$

and the SPME simplify to:

$$\langle k_a||O(M1,s)||k_b\rangle = \sqrt{\frac{3}{4\pi}}\langle j_a||\mathbf{s}||j_b\rangle\delta_{n_a,n_b}g_q^s\mu_N$$

$$= \sqrt{\frac{3}{4\pi}}(-1)^{l_a+j_a+3/2}\sqrt{(2j_a+1)(2j_b+1)}\left\{ \begin{array}{ccc} 1/2 & 1/2 & 1 \\ j_b & j_a & l_a \end{array} \right\}$$

$$\times\langle s||\mathbf{s}||s\rangle\delta_{l_a,l_b}\delta_{n_a,n_b}g_q^s\mu_N,$$

Electromagnetic multipole moments and transitions

where we have

$$< s||\mathbf{s}||s> = \sqrt{3/2},$$

and

$$< k_a||O(M1,l)||k_b> = \sqrt{\frac{3}{4\pi}} < j_a||\mathbf{l}||j_b> \delta_{n_a,n_b}\, g_q^l\, \mu_N$$

$$= \sqrt{\frac{3}{4\pi}}(-1)^{l_a+j_b+3/2}\sqrt{(2j_a+1)(2j_b+1)}\left\{ \begin{array}{ccc} l_a & l_b & 1 \\ j_b & j_a & 1/2 \end{array} \right\}$$

$$\times\langle l_a||\mathbf{l}||l_b\rangle\delta_{n_a,n_b}g_q^l\mu_N,$$

where

$$\langle l_a||\mathbf{l}||l_b\rangle = \delta_{l_a,l_b}\sqrt{l_a(l_a+1)(2l_a+1)}.$$

Thus the $M1$ operator connects only those orbitals which have the same n and l values.

β-decay

We will now focus on allowed β-decay. Suhonen's chapter 7 and Alex Brown's chapter 29 cover much of the material to be discussed on β-decay.

β-decay

The allowed beta decay rate W between a specific set of initial and final states is given by

$$W_{i,f} = (f/K_o)\left[g_V^2\, B_{i,f}(F_\pm) + g_A^2 B_{i,f}(GT_\pm)\right],$$

where f is dimensionless three-body phase-space factor which depends upon the beta-decay Q value, and K_o is a specific combination of fundamental constants

$$K_o = \frac{2\pi^3\hbar^7}{m_e^5c^4} = 1.8844\times 10^{-94}\mathrm{erg}^2\mathrm{cm}^6\mathrm{s}.$$

The $\pm$ signrefer to $\beta_\pm$ decay of nucleus (A_i, Z_i) into nucleus $(A_i, Z_i \mp 1)$. The weak-interaction vector (V) and axial-vector (A) coupling constants for the decay of neutron into a proton are denoted by g_V and g_A, respectively.

β-decay

The total decay rate for a given initial state is obtained by summing the partial rates over all final states

$$W = \sum_f W_{if},$$

with the branching fraction to a specific final state given by

$$b_{if} = \frac{W_{if}}{W}.$$

Beta decay lifetime are usually given in terms of the half-life with a total half-life of

$$T_{1/2} = \frac{\ln(2)}{W}.$$

The partial half-life for a particular final state will be denoted by $t_{1/2}$

$$t_{1/2} = \frac{T_{1/2}}{b_{if}}.$$

β-decay

Historically one combines the partial half-life for a particular decay with the calculated phase-space factor f to obtain an **ft** value given by

$$ft_{1/2} = \frac{C}{[B(F_\pm) + (g_A/g_V)^2 B(GT_\pm)]}$$

where

$$C = \frac{\ln(2)K_o}{(g_V)^2}.$$

β-decay

One often compiles the allowed beta decay in terms of a **logft** which stands for $\log_{10}$ of the $ft_{1/2}$ value.
The values of the coupling constants for Fermi decay, g_V, and Gamow-Teller decay, g_A are obtained as follows. For a $0^+ \to 0^+$ nuclear transition $B(GT) = 0$, and for a transition between $T = 1$ analogue states with $B(F) = 2$ we find

$$C = 2t_{1/2}f.$$

The partial half-lives and Q values for several $0^+ \to 0^+$ analogue transitions have been measured to an accuracy of about one part in 10000. With phase space factors one obtains

$$C = 6170(4)$$

This result, together with the value of K_o can be used to obtain g_V.

β-decay

At the quark level $g_V = -g_A$. But for nuclear structure we use the value obtained from the neutron to proton beta decay

$$|g_A/g_V| = 1.261(8).$$

β-decay

The operator for Fermi beta decay in terms of sums over the nucleons is

$$\mathcal{O}(F_\pm) = \sum_k t_{k\pm}.$$

The matrix element is

$$B(F) = |\langle f|T_\pm|i\rangle|^2,$$

where

$$T_\pm = \sum_k t_\pm$$

is the total isospin raising and lowering operator for total isospin constructed out of the basic nucleon isospin raising and lowering operators

$$t_-|n\rangle = |p\rangle \qquad t_-|p\rangle = 0,$$

and

$$t_+|p\rangle = |n\rangle, \quad t_+|n\rangle = 0.$$

β-decay

The matrix elements obey the triangle conditions $J_f = J_i$ ($\Delta J = 0$). The Fermi operator has $\pi_O = +1$, and thus the initial and final nuclear states must have $\pi_i \pi_f = +1$ for the matrix element to be nonzero under the parity transform.
When isospin is conserved the Fermi matrix element must obey the isospin triangle condition $T_f = T_i$ ($\Delta T = 0$), and the Fermi operator can only connect isobaric analogue states.

β-decay

For β_- decay
$$T_-|\omega_i, J_i, M_i, T_i, T_{zi}\rangle$$
$$= \sqrt{(T_i(T_i+1) - T_{zi}(T_{zi}-1)}|\omega_i, J_i, M_i, T_i, T_{zi}-1\rangle,$$
and
$$B(F_-) = |\langle\omega_f, J_f, M_f, T_f, T_{zi}-1|T_-|\omega_i, J_i, M_i, T_i, T_{zi}\rangle|^2$$
$$= [T_i(T_i+1) - T_{zi}(T_{zi}-1)]\delta_{\omega_f,\omega}\,\delta_{J_i,J_f}\,\delta_{M_i,M_f}\,\delta_{T_i,T_f}.$$

β-decay

For β_+ we have
$$B(F_+) = |\langle\omega_f, J_f, M_f, T_f, T_{zi}+1|T_+|\omega_i, J_i, M_i, T_i, T_{zi}\rangle|^2$$
$$= [T_i(T_i+1) - T_{zi}(T_{zi}+1)]\delta_{\omega_f,\omega}\,\delta_{J_i,J_f}\delta_{M_i,M_f}\delta_{T_i,T_f}.$$
For neutron-rich nuclei ($N_i > Z_i$) we have $T_i = T_{zi}$ and thus
$$B(F_-)(N_i > Z_i) = 2T_{zi} = (N_i - Z_i)\delta_{\omega_f,\omega}\delta_{J_i,J_f}\,\delta_{M_i,M_f}\,\delta_{T_i,T_f},$$
and
$$B(F_+)(N_i > Z_i) = 0.$$

β-decay

The reduced single-particle matrix elements are given by
$$\langle k_a, p||\sigma t_-||k_b, n\rangle = \langle k_a, n||\sigma t_+||k_b, p\rangle = 2\langle k_a||\mathbf{s}||k_b\rangle,$$
where the matrix elements of $\mathbf{s}$ are given by
$$\langle k_a||\mathbf{s}||k_b\rangle = \langle j_a||\mathbf{s}||j_b\rangle\delta_{n_a,n_b}$$
$$= (-1)^{l_a+j_a+3/2}\sqrt{(2j_a+1)(2j_b+1)}\begin{Bmatrix} 1/2 & 1/2 & 1 \\ j_b & j_a & l_a \end{Bmatrix}\langle s||\mathbf{s}||s\rangle\delta_{\ell_a,\ell_b}\delta_{n_a,n_b}$$
with
$$\langle s||\mathbf{s}||s\rangle = \sqrt{3/2}.$$

β-decay

The matrix elements of $\mathbf{s}$ has the selection rules δ_{ℓ_a,ℓ_b} and δ_{n_a,n_b}. Thus the orbits which are connected by the GT operator are very selective; they are those in the same major oscillator shell with the same ℓ value. The matrix elements such as $1s_{1/2} - 0d_{3/2}$ which have the allowed Δj coupling but are zero due to the $\Delta\ell$ coupling are called ℓ-**forbidden** matrix elements.

β-decay

Sum rules for Fermi and Gamow-Teller matrix elements can be obtained easily.
The sum rule for Fermi is obtained from the sum
$$\sum_f [B_{fi}(F_-) - B_{fi}(F_+)] = \sum_f [|\langle f|T_-|i\rangle|^2 - |\langle f|T_+|i\rangle|^2]$$
The final states f in the T_- matrix element go with the $Z_f = Z_i + 1$ nucleus and those in the T_+ matrix element to with the $Z_f = Z_i - 1$ nucleus. One can explicitly sum over the final states to obtain
$$\sum_f [\langle i|T_+|f\rangle\langle f|T_-|i\rangle - \langle i|T_-|f\rangle\langle f|T_+|i\rangle]$$
$$= \langle i|T_+T_- - T_-T_+|i\rangle = \langle i|2T_z|i\rangle = (N_i - Z_i).$$

β-decay

The sum rule for Gamow-Teller is obtained as follows
$$\sum_{f,\mu}|\langle f|\sum_k \sigma_{k,\mu}t_{k-}|i\rangle|^2 - \sum_{f,\mu}|\langle f|\sum_k \sigma_{k,\mu}t_{k+}|i\rangle|^2$$
$$= \sum_{f,\mu}\langle i|\sum_k \sigma_{k,\mu}t_{k+}|f\rangle\langle f|\sum_{k'}\sigma_{k',\mu}t_{k'-}|i\rangle$$
$$- \sum_{f,\mu}\langle i|\sum_k \sigma_{k,\mu}t_{k-}|f\rangle\langle f|\sum_{k'}\sigma_{k',\mu}t_{k'+}|i\rangle$$
$$= \sum_\mu \left[\langle i|\left(\sum_k \sigma_{k,\mu}t_{k+}\right)\left(\sum_{k'}\sigma_{k',\mu}t_{k'-}\right) - \left(\sum_k \sigma_{k,\mu}t_{k-}\right)\left(\sum_{k'}\sigma_{k',\mu}t_{k'}\right.\right.$$
$$= \sum_\mu \langle i|\sum_k \sigma^2_{k,\mu}[t_{k+}t_{k-} - t_{k-}t_{k+}]\,|i\rangle = 3\langle i|\sum_k [t_{k+}t_{k-} - t_{k-}t_{k+}]\,|i\rangle$$
$$= 3\langle i|T_+T_- - T_-T_+|i\rangle = 3\langle i|2T_z|i\rangle = 3(N_i - Z_i).$$

β-decay

We have used the fact that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$. When $k \neq k'$ the operators commute and cancel. Thus

$$\sum_f [B_{fi}(F_-) - B_{fi}(F_+)] = (N_i - Z_i),$$

and

$$\sum_f [B_{fi}(GT_-) - B_{fi}(GT_+)] = 3(N_i - Z_i).$$

The sum-rule for the Fermi matrix elements applies even when isospin is not conserved.

β-decay

For $N > Z$ we usually have $T_i = T_{zi}$ which means that $B(F_+) = 0$. For $N = Z (T_{zi} = 0)$ and $T_i = 0$ we get $B(F_+) = B(F_-) = 0$, and for $T_i = 1$ we have $B(F_+) = B(F_-) = 2$. Fermi transitions which would be zero if isospin is conserved are called isospin-forbidden Fermi transitions.
When $N > Z$ there are some situations where one has $B(GT_+) = 0$, and then we obtain $B(GT_-) = 3(N_i - Z_i)$. In particular for the β_- decay of the neutron we have $B(F_-) = 1$ and $B(GT_-) = 3$.

Core-polarization

We need to say something about so-called core-polarization effects. To do this, we have to introduce elements from many-body perturbation theory.
We assume here that we are only interested in the ground state of the system and expand the exact wave function in term of a series of Slater determinants

$$|\Psi_0\rangle = |\Phi_0\rangle + \sum_{m=1}^{\infty} C_m|\Phi_m\rangle,$$

where we have assumed that the true ground state is dominated by the solution of the unperturbed problem, that is

$$\hat{H}_0|\Phi_0\rangle = W_0|\Phi_0\rangle.$$

The state $|\Psi_0\rangle$ is not normalized, rather we have used an intermediate normalization $\langle\Phi_0|\Psi_0\rangle = 1$ since we have $\langle\Phi_0|\Phi_0\rangle = 1$.

Electromagnetic multipole moments and transitions

The Schroedinger equation is

$$\hat{H}|\Psi_0\rangle = E|\Psi_0\rangle,$$

and multiplying the latter from the left with $\langle\Phi_0|$ gives

$$\langle\Phi_0|\hat{H}|\Psi_0\rangle = E\langle\Phi_0|\Psi_0\rangle = E,$$

and subtracting from this equation

$$\langle\Psi_0|\hat{H}_0|\Phi_0\rangle = W_0\langle\Psi_0|\Phi_0\rangle = W_0,$$

and using the fact that the both operators $\hat{H}$ and $\hat{H}_0$ are hermitian results in

$$\Delta E = E - W_0 = \langle\Phi_0|\hat{H}_I|\Psi_0\rangle,$$

which is an exact result. We call this quantity the correlation energy.

Electromagnetic multipole moments and transitions

This equation forms the starting point for all perturbative derivations. However, as it stands it represents nothing but a mere formal rewriting of Schroedinger's equation and is not of much practical use. The exact wave function $|\Psi_0\rangle$ is unknown. In order to obtain a perturbative expansion, we need to expand the exact wave function in terms of the interaction $\hat{H}_I$.
Here we have assumed that our model space defined by the operator $\hat{P}$ is one-dimensional, meaning that

$$\hat{P} = |\Phi_0\rangle\langle\Phi_0|,$$

and

$$\hat{Q} = \sum_{m=1}^{\infty} |\Phi_m\rangle\langle\Phi_m|.$$

Electromagnetic multipole moments and transitions

We can thus rewrite the exact wave function as

$$|\Psi_0\rangle = (\hat{P} + \hat{Q})|\Psi_0\rangle = |\Phi_0\rangle + \hat{Q}|\Psi_0\rangle.$$

Going back to the Schrödinger equation, we can rewrite it as, adding and a subtracting a term $\omega|\Psi_0\rangle$ as

$$\left(\omega - \hat{H}_0\right)|\Psi_0\rangle = \left(\omega - E + \hat{H}_I\right)|\Psi_0\rangle,$$

where ω is an energy variable to be specified later.

We assume also that the resolvent of $\left(\omega-\hat{H}_0\right)$ exits, that is it has an inverse which defined the unperturbed Green's function as

$$\left(\omega-\hat{H}_0\right)^{-1}=\frac{1}{\left(\omega-\hat{H}_0\right)}.$$

We can rewrite Schroedinger's equation as

$$|\Psi_0\rangle=\frac{1}{\omega-\hat{H}_0}\left(\omega-E+\hat{H}_I\right)|\Psi_0\rangle,$$

and multiplying from the left with $\hat{Q}$ results in

$$\hat{Q}|\Psi_0\rangle=\frac{\hat{Q}}{\omega-\hat{H}_0}\left(\omega-E+\hat{H}_I\right)|\Psi_0\rangle,$$

which is possible since we have defined the operator $\hat{Q}$ in terms of the eigenfunctions of $\hat{H}$.

These operators commute meaning that

$$\hat{Q}\frac{1}{\left(\omega-\hat{H}_0\right)}\hat{Q}=\hat{Q}\frac{1}{\left(\omega-\hat{H}_0\right)}=\frac{\hat{Q}}{\left(\omega-\hat{H}_0\right)}.$$

With these definitions we can in turn define the wave function as

$$|\Psi_0\rangle=|\Phi_0\rangle+\frac{\hat{Q}}{\omega-\hat{H}_0}\left(\omega-E+\hat{H}_I\right)|\Psi_0\rangle.$$

This equation is again nothing but a formal rewrite of Schrödinger's equation and does not represent a practical calculational scheme. It is a non-linear equation in two unknown quantities, the energy E and the exact wave function $|\Psi_0\rangle$. We can however start with a guess for $|\Psi_0\rangle$ on the right hand side of the last equation.

The most common choice is to start with the function which is expected to exhibit the largest overlap with the wave function we are searching after, namely $|\Phi_0\rangle$. This can again be inserted in the solution for $|\Psi_0\rangle$ in an iterative fashion and if we continue along these lines we end up with

$$|\Psi_0\rangle=\sum_{i=0}^{\infty}\left\{\frac{\hat{Q}}{\omega-\hat{H}_0}\left(\omega-E+\hat{H}_I\right)\right\}^i|\Phi_0\rangle,$$

for the wave function and

$$\Delta E=\sum_{i=0}^{\infty}\langle\Phi_0|\hat{H}_I\left\{\frac{\hat{Q}}{\omega-\hat{H}_0}\left(\omega-E+\hat{H}_I\right)\right\}^i|\Phi_0\rangle,$$

which is now a perturbative expansion of the exact energy in terms of the interaction $\hat{H}_I$ and the unperturbed wave function $|\Psi_0\rangle$.

In our equations for $|\Psi_0\rangle$ and ΔE in terms of the unperturbed solutions $|\Phi_i\rangle$ we have still an undetermined parameter ω and a dependecy on the exact energy E. Not much has been gained thus from a practical computational point of view.
In Brillouin-Wigner perturbation theory it is customary to set $\omega=E$. This results in the following perturbative expansion for the energy ΔE

$$\Delta E=\sum_{i=0}^{\infty}\langle\Phi_0|\hat{H}_I\left\{\frac{\hat{Q}}{\omega-\hat{H}_0}\left(\omega-E+\hat{H}_I\right)\right\}^i|\Phi_0\rangle=$$

$$\langle\Phi_0|\left(\hat{H}_I+\hat{H}_I\frac{\hat{Q}}{E-\hat{H}_0}\hat{H}_I+\hat{H}_I\frac{\hat{Q}}{E-\hat{H}_0}\hat{H}_I\frac{\hat{Q}}{E-\hat{H}_0}\hat{H}_I+\dots\right)|\Phi_0\rangle.$$

$$\Delta E=\sum_{i=0}^{\infty}\langle\Phi_0|\hat{H}_I\left\{\frac{\hat{Q}}{\omega-\hat{H}_0}\left(\omega-E+\hat{H}_I\right)\right\}^i|\Phi_0\rangle=$$

$$\langle\Phi_0|\left(\hat{H}_I+\hat{H}_I\frac{\hat{Q}}{E-\hat{H}_0}\hat{H}_I+\hat{H}_I\frac{\hat{Q}}{E-\hat{H}_0}\hat{H}_I\frac{\hat{Q}}{E-\hat{H}_0}\hat{H}_I+\dots\right)|\Phi_0\rangle.$$

This expression depends however on the exact energy E and is again not very convenient from a practical point of view. It can obviously be solved iteratively, by starting with a guess for E and then solve till some kind of self-consistency criterion has been reached.
Actually, the above expression is nothing but a rewrite again of the full Schrödinger equation.

Defining $e=E-\hat{H}_0$ and recalling that $\hat{H}_0$ commutes with $\hat{Q}$ by construction and that $\hat{Q}$ is an idempotent operator $\hat{Q}^2=\hat{Q}$. Using this equation in the above expansion for ΔE we can write the denominator

$$\hat{Q}\frac{1}{\cdots}=$$

Inserted in the expression for ΔE leads to

$$\Delta E=\langle\Phi_0|\hat{H}_I+\hat{H}_I\hat{Q}\frac{1}{E-\hat{H}_0-\hat{Q}\hat{H}_I\hat{Q}}\hat{Q}\hat{H}_I|\Phi_0\rangle.$$

In RS perturbation theory we set $\omega=W_0$ and obtain the following expression for the energy difference

$$\Delta E=\sum_{i=0}^{\infty}\langle\Phi_0|\hat{H}_I\left\{\frac{\hat{Q}}{W_0-\hat{H}_0}\left(\hat{H}_I-\Delta E\right)\right\}^i|\Phi_0\rangle=$$

$$\langle\Phi_0|\left(\hat{H}_I+\hat{H}_I\frac{\hat{Q}}{W_0-\hat{H}_0}(\hat{H}_I-\Delta E)+\hat{H}_I\frac{\hat{Q}}{W_0-\hat{H}_0}(\hat{H}_I-\Delta E)\frac{\hat{Q}}{W_0-\hat{H}_0}(\hat{H}\right.$$

Electromagnetic multipole moments and transitions

Recalling that $\hat{Q}$ commutes with $\hat{H}_0$ and since ΔE is a constant we obtain that

$$\hat{Q}\Delta E|\Phi_0\rangle = \hat{Q}\Delta E|\hat{Q}\Phi_0\rangle = 0.$$

Inserting this results in the expression for the energy results in

$$\Delta E = \langle\Phi_0|\left(\hat{H}_I + \hat{H}_I\frac{\hat{Q}}{W_0-\hat{H}_0}\hat{H}_I + \hat{H}_I\frac{\hat{Q}}{W_0-\hat{H}_0}(\hat{H}_I-\Delta E)\frac{\hat{Q}}{W_0-\hat{H}_0}\hat{H}_I + \right.$$

Electromagnetic multipole moments and transitions

We can now this expression in terms of a perturbative expression in terms of $\hat{H}_I$ where we iterate the last expression in terms of ΔE

$$\Delta E = \sum_{i=1}^{\infty}\Delta E^{(i)}.$$

We get the following expression for $\Delta E^{(i)}$

$$\Delta E^{(1)} = \langle\Phi_0|\hat{H}_I|\Phi_0\rangle,$$

which is just the contribution to first order in perturbation theory,

$$\Delta E^{(2)} = \langle\Phi_0|\hat{H}_I\frac{\hat{Q}}{W_0-\hat{H}_0}\hat{H}_I|\Phi_0\rangle,$$

which is the contribution to second order.

Electromagnetic multipole moments and transitions

$$\Delta E^{(3)} = \langle\Phi_0|\hat{H}_I\frac{\hat{Q}}{W_0-\hat{H}_0}\hat{H}_I\frac{\hat{Q}}{W_0-\hat{H}_0}\hat{H}_I\Phi_0\rangle - \langle\Phi_0|\hat{H}_I\frac{\hat{Q}}{W_0-\hat{H}_0}\langle\Phi_0|\hat{H}_I|\Phi_0\rangle$$

being the third-order contribution.

Learning outcomes

I hope this is not the case

Topics we have covered this year

- Single-particle properties and mean-field and relation to data
- How to set up basis states in second quantization and find expectation values
- Angular momentum properties and the Wigner-Eckart theorem
- Short survey of properties of nuclear forces
- The nuclear shell model
- And how to relate a shell-model calculation to decays and properties of decays.

Final presentation

- Introduction with motivation
- Explain an eventual experimental set up
- Give a short overview of the theory employed and how it relates to the analysis of eventual data
- Present and discuss your results
- Summary, conclusions and perspectives
- Anything else you think is important. Useful to have backup slides

In total your talk should have a duration of 20-25 minutes, but longer is also ok.

What? Me worry?

Best wishes to you all and thanks so much for your efforts this semester